Second Grade Fluid performing Sinusoidal Motion in an Infinite Cylinder

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Abstract—The purpose of this work is to obtain some new results for non-Newtonian fluid of type second grade performing sinusoidal motion. Exact solution of the velocity field and shear stress corresponding to second grade in an infinite cylinder are obtained by applying Laplace and Hankel transforms. Generalized function G.,.,.(.,t) and Bessel functions have been used to write solutions in the form of series. Similar solutions for Newtonian fluids performing the same motion are recovered as a limiting case.

Index Terms—Exact solution, Second grade fluid, Shear stress.

1 Introduction

Investigation of exact solutions for non-Newtonian fluids has gained considerable importance among researchers for many reasons. Exact solutions provide standard for accuracy checking of approximation methods which may be empirical or numerical. Although, by applying some computer techniques complete integration of equations of motion for non-Newtonian fluids is possible, but accuracy can be established by comparing with an exact solution. Exact solutions are also important to verify and test numerical schemes which are developed to study problems concerning complex and unsteady flow. There is a great diversity in the physical structure of non-Newtonian fluids. [3,7], many models have been proposed to describe the response characteristics of fluids which cannot be described by classical Navier-Stokes model of equation of motion . The fluids, either natural or synthetic are mixtures of different materials such as particles, water, red cell, oils and long chain molecules, their viscosity can be determined by elongational and time dependent effects. Viscosity function generally varies non-linearly with shear rate. Fluids can be treated as viscoelastic [9, 11]. Fractional calculus has much encountered success in describing elastic behaviour of fluids [2,5]. Many models have been proposed to describe viscoelastic behaviour of non-Newtonian fluids. Generally, non-Newtonian fluids are classified into three types a) Differential type b) the rate type c) the integral type. Among these the differential type fluids have gained much popularity. A subclass of differential type fluid is second grade fluid which has been studied extensively by researchers.

A starting solution of the motion of second grade fluid due to longitudinal and torsional oscillation of cylinder have been established in [4].Some solutions for different oscillatory motions of non-Newtonian fluid have been obtained by T. Hayat [6]. A. Mahmood [8] found exact solution of viscoelastic non-Newtonian fluid corresponding to longitudinal oscillatory flow whereas D. Vieru [10] found exact solution for the motion of Maxwell fluid due to longitudinal and torsional oscillations in an infinite cylinder by means of Laplace transform. M.Athar [1] solved Taylor-Couette flow equations of generalized second grade fluid with the help of Laplace and Hankel transforms.

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2 Governing Equations

In this paper we consider the velocity V and the extra stress S of the form

$$V = V(r,t) = \omega(r,t)e_\theta$$
$$S = S(r,t)$$

Where e_θ is the unit vector in the θ direction of the cylindrical coordinate system. At $t = 0$ we have

$$\omega(r,0) = 0$$

The governing equations corresponding to such motion of ordinary second grade fluid are

$$\tau(r,t) = (\mu + \alpha_1 \frac{\partial}{\partial t})(\frac{\partial}{\partial r} - \frac{1}{r})\, \omega(r,t) \tag{1}$$

$$\frac{\partial\omega(r,t)}{\partial t} = (\vartheta + \frac{\alpha\,\partial}{\partial t})(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2})\, \omega(r,t) \tag{2}$$

$$\alpha_1 = \alpha\rho \text{ is material constant}$$

$$\vartheta = \frac{\mu}{\rho} \text{ Kinematic viscosity}$$

$$\tau(r,t) = Sr_\theta(r,t)$$

3 Flow with Shear on the Boundary

$$\omega(r,0) = 0 \qquad \text{Where } r\epsilon(0,R] \qquad (3)$$

$$\tau(R,t) = (\mu + \alpha_1 \frac{\partial}{\partial t})(\frac{\partial}{\partial r} - \frac{1}{r})\,\omega(R,t)I_{(R=r)}$$

$$A\sin(\omega t) \qquad t > 0 \qquad (4)$$

4 Calculation of Velocity Field

Applying Laplace transform to (2)

$$q\overline{\omega}(r,q) = (\vartheta + \alpha q)(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2})\overline{\omega}(r,q) \qquad (5)$$

Applying Laplace transform to (4)

$$\overline{\tau}(r,q) = (\mu + \alpha_1 q)|(\frac{\partial}{\partial r} - \frac{1}{r})\,\overline{\omega}(r,q)|_{R=r)}$$

$$= A\frac{\omega}{\omega^2+q^2} \qquad (6)$$

Applying Hankel transform to equation (5)

$$q\overline{\omega}_H(r_n,q) = (\vartheta + \alpha q)\int_0^R r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{\omega}(r,q)J_1(rr_n)dr \qquad (7)$$

Where

$$\int_0^R r\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2}\right)\overline{\omega}(r,q)J_1(rr_n)dr$$

$$= RJ_1(Rr_n)|\frac{\partial\overline{\omega}(r,q)}{\partial r} - \frac{1}{r}\overline{\omega}(r,q)|_0^R - r_n^2\,\overline{\omega}_H(r_n,q)$$

Using this result in (7)

$$\overline{\omega}_H(r_n,q) = \frac{RJ_1(Rr_n)A\omega}{\omega^2+q^2}\left[\frac{(\vartheta+\alpha q)}{(\mu+\alpha_1 q)(q+\alpha r_n^2 q+\vartheta r_n^2)}\right] \qquad (8)$$

Separating into two parts

We have

$$\overline{\omega}_{H1}(r_n,q) = \frac{RJ_1(Rr_n)A\omega}{\mu r_n^2(\omega^2+q^2)} \qquad (9)$$

And

$$\overline{\omega}_{H2}(r_n,q) = -\frac{RJ_1(Rr_n)A\omega}{\mu r_n^2(\omega^2+q^2)}\bullet\frac{q(1+\alpha r_n^2)}{(q+\alpha r_n^2 q+\vartheta r_n^2)} \qquad (10)$$

Applying Hankel inverse transform to (9) and (10)

We get

$$\overline{\omega}_1(r,q) = \frac{A\omega}{2R^2\mu\,(\omega^2+q^2)}\bullet r^3 \qquad (11)$$

And

$$\overline{\omega}_2(r,q) = -\frac{2}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)A\omega}{J_1(Rr_n)}\bullet\frac{q(1+\alpha r_n^2)}{\mu r_n^2(\omega^2+q^2)(q+\alpha r_n^2 q+\vartheta r_n^2)} \qquad (12)$$

Applying Laplace inverse transform to (11) and (12)

Where

$$\omega_1(r,t) = \frac{Ar^3}{2R^2\mu}\sin(\omega t) \qquad (13)$$

And

$$\omega_2(r,t) = -\frac{2}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)A}{J_1(Rr_n)\mu r_n^2}\,x$$

$$[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t \sin(\omega s)G_{0,-K,K+1}(-\alpha r_n^2, t-s)ds +$$

$$\alpha r_n^2\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t \sin(\omega s)G_{0,-K,K+1}(-\alpha r_n^2, t-s)ds \qquad (14)$$

Adding (13) and (14) we get velocity field as

$$\omega(r,t) = \frac{Ar^3}{2R^2\mu}\sin(\omega t) - \frac{2}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)A}{J_1(Rr_n)\mu r_n^2}\,x$$

$$[\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t \sin(\omega s)G_{0,-K,K+1}(-\alpha r_n^2, t-s)ds +$$

$$\alpha r_n^2\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k\int_0^t \sin(\omega s)G_{0,-K,K+1}(-\alpha r_n^2, -s)ds] \qquad (15)$$

5 Calculation of Shear Stress

$$\tau(r,t) = (\mu + \alpha_1 \frac{\partial}{\partial t})(\frac{\partial}{\partial r} - \frac{1}{r})\,\omega(r,t)$$

Applying Laplace transform we get

$$\overline{\tau}(r,q) = (\mu + \alpha_1 q)(\frac{\partial}{\partial r} - \frac{1}{r})\,\overline{\omega}(r,q) \qquad (16)$$

Breaking equation (8) in different way, we get

$$\overline{\omega}_{1H} = \frac{RJ_1(Rr_n)A\omega}{r_n^2}\left[\frac{1}{(\omega^2+q^2)(\mu+\alpha_1 q)}\right] \qquad (17)$$

$$\overline{\omega}_{2H}(r_n,q) =$$

$$\frac{RJ_1(Rr_n)A\omega}{r_n^2} \times \left[\frac{q}{(\omega^2+q^2)(\mu+\alpha_1 q)(q+\alpha r_n^2 q+\vartheta r_n^2)}\right] \qquad (18)$$

Applying Hankel inverse transform to (17), (18) and putting in (16) and applying Laplace inverse transform using convolution theorem we get.

$$\tau(r,t) = \frac{Ar^2}{R^2}\sin(\omega t) + \frac{2}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)A}{r_n^2 J_1(Rr_n)} \times$$

$$\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{0,-K,K+1}(-\alpha r_n^2, t-s)ds$$

6 Limiting Case

6.1 Newtonian

$\alpha \to 0$ and $\alpha_1 \to 0$ In velocity field and shear stress enable us to get corresponding Newtonian velocity and shear stress performing same motion.

$$\omega(r,t) = \frac{Ar^3}{2R^2\mu} - \frac{2}{R}\sum_{n=1}^{\infty}\frac{J_1(rr_n)A}{J_1(Rr_n)\mu r_n^2} \times$$

$$\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{0,-K,K+1}(0, t-s)ds$$

And

$$\tau(r,t) = \frac{Ar^2}{R^2}\sin(\omega t) + \frac{2}{R}\sum_{n=1}^{\infty}\frac{J_2(rr_n)A}{r_n^2 J_1(Rr_n)} \times$$

$$\sum_{k=0}^{\infty}(-\vartheta r_n^2)^k \int_0^t \sin(\omega s) G_{0,-K,K+1}(0, t-s)ds$$

7 Conclusion

The exact solution of the velocity field and associated shear stress corresponding to second grade fluid in an infinite cylinder. Performing sinusoidal motion are obtained by applying Laplace transform and Hankel transform. The solutions have been written in series form using generalized G.,..(.,t) function and Bessel function. Similar solutions for Newtonian fluid performing the same motion are obtained.

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